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Short proofs of some theorems in approximation theory

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Abstract

New proofs, shorter than previous ones, of three theorems from approximation theory are given. The Schwarz's lemma is a key tool.

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1. Introduction

In this paper, we will give new proofs of three theorems from approximation theory. All the theorems are well known and proven long ago. The proofs presented here, however, are shorter than previous ones and use a common approach.

Let K be a compact subset of the complex plane. Further, let $f \in C(K)$, the set of all continuous complex-valued functions on K with the supremum norm. It is well known that the degree of approximation of f in $C(K)$ by polynomials of degree $\leqslant N$ satisfies the following formula:

$$E_N(f,K) = \sup_{\mu} \left| \int_K f(z)\, d\mu(z) \right|. \tag{1}$$

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The supremum is taken over all complex Borel measures μ on K of total variation 1 satisfying

$$\int_K z^k \, d\mu(z) = 0, \quad k = 0, 1, \dots, N. \tag{2}$$

It is a necessary condition for the numbers $E_N(f, K)$ to tend to zero as N tends to infinity, that f is holomorphic in the interior of K. We will study functions f holomorphic in a set containing K.

In the rest of this paper, U will denote the open unit disc in the complex plane and $\overline{U}$ its closure.

2. Proofs

For earlier proofs of the theorems in this section, see [2].

Theorem 1. *Let f be holomorphic in ρU, $\rho > 1$, and bounded by M in modulus. Let N be a natural number. Then*

$$E_N(f, \overline{U}) \leqslant \frac{M}{\rho^{N+1}}.$$

Proof (New). Take a μ as in (1). It is sufficient to show that

$$\left| \int_{\overline{U}} f(z) \, d\mu(z) \right| \leqslant \frac{M}{\rho^{N+1}}. \tag{3}$$

Define g by

$$g(\zeta) = \int_{\overline{U}} f(\zeta z) \, d\mu(z), \quad \zeta \in \rho U.$$

Obviously, $|g(\zeta)| \leqslant M$, and it is easy to show that g is holomorphic. If we insert ζz into the power series expansion of f around 0 we get

$$f(\zeta z) = \sum_{k=0}^{\infty} c_k \zeta^k z^k.$$

The series converges uniformly for $z \in \overline{U}$ and $|\zeta|$ small. Term-by-term integration with respect to z and μ together with (2) shows that g has a zero of order $\geqslant N + 1$ at $\zeta = 0$. Since $\int_{\overline{U}} f(z) \, d\mu(z) = g(1)$, it is sufficient to estimate g. But this can be done with an extended version of the Schwarz's lemma, in which the function has a multiple zero at the origin. We have $|g(1)| \leqslant M/\rho^{N+1}$, and (3) follows. $\square$

It is easy to extend the proof to polydiscs:

Theorem 2. *Let $K \subset \mathbf{C}^n$ be compact and have the property that $\xi \in U \Rightarrow \xi K \subset K^o$, the interior of K. Let f be holomorphic in ρK^o, $\rho > 1$, and bounded by M in modulus. Then*

$$E_N(f, K) \leqslant \frac{M}{\rho^{N+1}}.$$

Proof (New). First we observe that $K = \rho\rho^{-1}K \subset \rho K^o$ and that the origin in $\mathbf{C}^n$, $\mathbf{0}$, belongs to ρK^o.

Let μ be a Borel measure on K of total variation 1 which is such that the integrals of polynomials of degree $\leqslant N$ vanish. For $\zeta \in \rho U$ define

$$g(\zeta) = \int_K f(\zeta \mathbf{z}) d\mu(\mathbf{z}) = \int_K f(\zeta z_1, \zeta z_2, \ldots, \zeta z_n)\, d\mu(z_1, z_2, \ldots, z_n).$$

The integrand is defined since

$$|\zeta| < \rho \Rightarrow \zeta \mathbf{z} = \rho(\zeta/\rho)\mathbf{z} \in \rho U K \subset \rho K^o.$$

It is easy to show that g is holomorphic and that $|g(\zeta)| \leqslant M$. If we insert $\zeta z_1, \zeta z_2, \ldots, \zeta z_n$ into the power series expansion of f around $\mathbf{0}$, we get

$$f(\zeta \mathbf{z}) = \sum_{\mathbf{k} \in \mathbf{N}^n} c_{\mathbf{k}} \zeta^{|\mathbf{k}|} \mathbf{z}^{\mathbf{k}},$$

where $\mathbf{k} = (k_1, k_2, \ldots, k_n)$ are multiindices, $\mathbf{z}^{\mathbf{k}} = \prod_{i=1}^n z_i^{k_i}$ and $|\mathbf{k}| = \sum_{i=1}^n k_i$. The series converges absolutely and uniformly for $\mathbf{z} \in K$ and $|\zeta|$ small. Term-by-term integration together with the fact that $\int_K \mathbf{z}^{\mathbf{k}}\, d\mu(\mathbf{z}) = 0$ if $|\mathbf{k}| \leqslant N$ shows that g has a zero of order at least $N+1$ at $\zeta = 0$. Now

$$\left| \int_K f(\mathbf{z})\, d\mu(\mathbf{z}) \right| = |g(1)| \leqslant \frac{M}{\rho^{N+1}}$$

by the Schwarz's lemma. □

The last theorem was first proved by Babenko [1]. Here a derivative of f is bounded. See also [2, p. 126; 3].

Theorem 3. *Let f be holomorphic in ρU, $\rho > 1$, and let the derivative of order p of f be bounded by M in modulus. If N is a natural number $\geqslant p-1$, we have*

$$E_N(f, \overline{U}) \leqslant \frac{M\rho^{p-N-1}}{\prod_{v=0}^{p-1} (N+1-v)}.$$

Proof (New). Take a μ and define g as in the first proof. Again, g has a zero of order at least $N+1$ at $\zeta = 0$. We have

$$g^{(p)}(\zeta) = \int_{\overline{U}} z^p f^{(p)}(\zeta z)\, d\mu(z).$$

Obviously, $g^{(p)}$ is bounded by M in modulus and has a zero of order at least $N+1-p$ at $\zeta=0$. Hence

$$|g^{(p)}(t)| \leqslant M\left(\frac{t}{\rho}\right)^{N+1-p}, \quad 0 \leqslant t \leqslant 1$$

by the Schwarz's lemma. If we integrate this relation p times and use the fact that $g^{(v)}(0)=0,\ v=0,1,\ldots,p-1$, we will find that $|g(1)| = |\int_{\overline{U}} f(z)\,d\mu(z)|$ does not exceed the right-hand side of the inequality in the theorem. □

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